Open-endedness, schemas and ontological commitment*

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Abstract

Second-order axiomatizations of certain important mathematical theories – such as arithmetic and real analysis – can be shown to be categorical. Categoricity implies semantic completeness, and semantic completeness in turn implies determinacy of truth-value. Second-order axiomatizations are thus appealing to realists as they sometimes seem to offer support for the realist thesis that mathematical statements have determinate truth-values. The status of second-order logic is a controversial issue, however. Worries about ontological commitment have been influential in the debate. Recently, Vann McGee has argued that one can get some of the technical advantages of second-order axiomatizations – categoricity, in particular – while walking free of worries about ontological commitment. In so arguing he appeals to the notion of an open-ended schema – a schema that holds no matter how the language of the relevant theory is extended. Contra McGee, we argue that second-order quantification and open-ended schemas are on a par when it comes to ontological commitment.

1 Categoricity and determinacy of truth-value

A theory T is *categorical* if, and only if, any two models M_1 and M_2 of T are isomorphic. That is, a theory T is categorical if and only if there is a bijection f between any two models M_1 and M_2 that preserves internal structure.[1] Provided that T is categorical, it is straightforward to show that T is also semantically complete – which is to say that, for for any sentence φ in the language of T, either $T \vDash \varphi$ or $T \vDash \neg\varphi$. Now, let T be a theory, and let $\mathbf{M}$ be the class of models of T. Then a sentence φ in the language of T is *determinately true* if and only if φ is true in every member of $\mathbf{M}$, and φ is *determinately false* if and only if $\neg\varphi$ is true in every member of $\mathbf{M}$.

What it is crucial to note for our current purposes is that, if T is a categorical theory, every sentence S in the language of T has a determinate truth-value. That is, if T is categorical, every statement S is either true in all models of T or false in all models of T. This is because categoricity implies semantic completeness, and semantic completeness implies determinacy of truth-value. (Both implications can be straightforwardly verified.)

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[1] This is merely an informal gloss on the notion of an isomorphism. However, for our present purposes it will suffice. For a full definition of the notion, cf., e.g., Enderton [4]. The notion of a theory relied on here is standard: Let A be the axioms of a theory T. The theory of T is the closure of A under semantic consequence, i.e. $T = \{\alpha : A \vDash \alpha\}$.

A truth-value realist with respect to a given discourse D embraces the following thesis:

(TVR) Every sentence of D is either determinately true or determinately false.

In the literature, some authors have taken categoricity to be philosophically significant because of its conceptual link with determinacy of truth-value. Since it implies semantic completeness – which in turn implies determinacy of truth-value – categoricity appears to give the truth-value realist exactly what she is after: a way of supporting (TVR).[2]

First-order theories cannot support (TVR), for well-known reasons. First-order theories that are strong enough to represent elementary arithmetic (and are recursively enumerable and assumed to be consistent) cannot be semantically complete, because, by Gödel's second incompleteness theorem, there is a statement ψ such that neither $T \vDash \psi$ nor $T \vDash \neg\psi$. This statement will be neither determinately true nor determinately false. However, if the background logic is changed from being first-order to second-order, the prospects of supporting (TVR) grow considerably brighter. Second-order Peano arithmetic (PA^2), second-order real analysis (RA^2) and second-order ZFC with urelements ($ZFCU^2$) + Urelement Set Axiom can all be shown to be categorical. It would thus seem that, at least for some theories, the realist can get determinacy of truth-value – a substantial philosophical thesis – by adopting a second-order axiomatization as her favoured formulation of the relevant theory.[3]

2 Second-order logic and ontological commitment

Can the truth-value realist cash in on the promises of second-order logic? The literature invites caution in answering this question, as the status of second-order logic has been – and still is – a hotly debated topic. One of the main sources of concern is ontological commitments incurred by second-order quantification. There is really a cluster of similar – yet distinct – worries about ontological commitment, but the one that appears to motivate McGee's adoption of open-ended schemas can be put as follows:

> Consider second-order axiomatizations of mathematical theories – arithmetic, say, i.e. the theory of the natural numbers. The axioms of second-order arithmetic include the induction axiom:

[2] What we call 'truth-value realism' is referred to as 'realism in truth-value' and 'realism about truth' in respectively Shapiro [15] and McGee [12]. The conceptual link between categoricity and truth-value realism has been contested by some (e.g. Field [7] and [6]). However, for the purposes of this paper, it will be assumed that the link does obtain since the objective is to discuss the claimed philosophical advantages of open-ended schemas over second-order logic as a means to achieve categoricity and determinacy of truth-value.

[3] For the categoricity of second-order arithmetic and real analysis case, see Shapiro [15], pp. 82–84. For $ZFCU^2$ + Urelement Set Axiom, consult McGee [12]. The Urelement Set Axioms states that all the urelements (i.e. non-sets) form a set. Note that categoricity only holds of the pure sets, i.e. sets that contain no non-sets. It should also be noted that in order for the proof of categoricity to go through it needs to be assumed that the range of the first-order quantifiers is unrestricted. Second-order ZFC (ZFC^2) is "only" quasi-categorical, i.e. for any two models M and N of the theory, one is isomorphic to an initial segment of the other (this result a variation of a result of Zermelo [16]).

To achieve categoricity (and quasi-categoricity) for second-order axiomatizations it is crucial that the semantics be taken to be standard rather than Henkin semantics (see Shapiro [15] for details). In what follows, we use 'second-order quantification' and 'second-order logic' to refer to their standard model-theoretic conceptions.

$$(\forall X)((X\mathbf{0} \wedge (\forall x)(Xx \rightarrow Xs(x))) \rightarrow (\forall x)Xx)$$

> The usual way of doing semantics for second-order logic is to take the second-order variables to range over the arbitrary subsets of the first-order domain.[4] Hence, adopting the second-order axiomatization of arithmetic not only commits one to numbers – the proper subject-matter of arithmetic – but to numbers *and* sets, or classes, of numbers. In general, second-order quantification commits one not just to whatever range of entities are in the first-order domain, but additionally to the collection of the arbitrary subsets of these entities – to serve as values of the second-order variables. The extra ontological commitments of second-order quantification are "unsavory", because they concern entities that are not properly speaking part of the subject-matter of the target theory – thus entities which an axiomatization of the theory should not commit one to.[5]

The criterion of ontological commitment operative in this line of reasoning is that of Quine. That is, it is assumed that to be is to be the value of a (bound) variable.[6] There are, of course, those who are critical of Quinean orthodoxy.[7] What is relevant here, however, is that McGee is not among the critical voices,[8] and so we have allowed ourselves to follow him in his reliance on the Quinean criterion of ontological commitment. (Strictly speaking, we will be relying on a slightly modified version of Quine's criterion, adapted – reasonably, as we think – to render it applicable to a wider range of cases, including the one at hand. More on this in section 4.)

One might think it relevant to distinguish between two kinds of considerations concerning ontological commitment: (i) what *types* of entity a theory is committed to, and (ii) what *scale* of commitment is incurred by the theory. In light of this distinction, there are two ways in which an axiomatization of a given theory might be thought to involve "unsavory ontological commitments". First, it might be thought to do so if it brings on a commitment to a type of entity which is not properly speaking part of the subject-matter of the relevant theory – however many, or few, entities of such a type the axiomatization commits one to. Second, an axiomatization might be found troublesome if it commits to "too many" entities in some way or other.

Whichever way the worry about unsavory ontological commitments is construed, second-order quantification is in trouble. If it is understood in the first way, the source of trouble is the mere commitment to sets of whatever entities are in the first-order domain; if the worry is understood in the second way, it is the large scale of this commitment. Second-order quantification brings on a commitment to the arbitrary subsets of the first-order domain. Where κ is the cardinality of the first-order domain, there are 2^κ-many subsets of the first-order domain. By Cantor's Theorem, $2^\kappa > \kappa$. Second-order quantification thus brings on a grand scale ontological commitment by incurring a commitment to a number of sets of whatever entities are in the first-order domain which is greater the number of objects in the first-order domain itself. In the case of arithmetic, for example, adopting the second-order axiomatization incurs a commitment to not just $\aleph_0$-many numbers, but additionally to $2^{\aleph_0}$-many sets of these.

[4] See Shapiro [15], Section 3.3.

[5] This strikes us as a reasonable gloss on what McGee has in mind. See [12], p. 60.

[6] See e.g. Quine [14].

[7] As Boolos put it nicely, "the difficulty with the question [concerning ontological commitment] is that the ground rules for answering it appear to have been laid down by Professor Quine." (Boolos [3], p. 76).

[8] McGee [12], p. 60. Lavine also appears to adopt it – see [8], p. 227.

3 McGee on schemas and open-endedness

In recent work, McGee has argued that categoricity results can be had for central mathematical theories without raising the kind of worry about set-theoretic commitments presented in the previous section. The way to avoid trouble is to rely on *open-ended schemas.* If McGee is right, such schemas offer a considerable improvement over second-order quantification from a philosophical point of view.[9]

To see what the notion of an open-ended schema amounts to let us first turn to first-order arithmetic (PA^1). PA^1 includes the axiom schema of induction:

$$(\Phi(\mathbf{0}) \wedge (\forall x)(\Phi(x) \rightarrow \Phi(s(x)))) \rightarrow (\forall x)\Phi(x)$$

i.e. for any open sentence of the language Φ, if Φ holds of zero and Φ holds of the successor of any natural number provided Φ holds of that natural number itself, then Φ holds of all natural numbers.

It is important to make clear that the schema itself is not an axiom of the theory. There is a simple reason for this: it cannot be written down as a single sentence in the language of the theory. However, what *is* part of the theory is each of the individual axioms generated by substituting an open formula into the schema.

The schema allows us to substitute *any* open sentence of PA^1 to obtain an individual axiom. In this sense, the schema is quite liberal. Yet, it does come with a substantive, principled restriction: for Φ we are only allowed to substitute an open sentence *in the language of first-order arithmetic*, i.e. the language of first-order logic supplemented by '**0**' and 's'.

Open-ended schema arithmetic is obtained by abandoning the restriction just high-lighted, i.e. by assuming the induction schema to hold no matter how the language is extended. This is the sense in which the schema is *open-ended.* According to McGee, what we learn when we learn the language of arithmetic is, rationally reconstructed, open-ended schema arithmetic: "Our understanding of the language of arithmetic is such that we anticipate that the Induction Axiom Schema, like the laws of logic, will persist through all such changes [i.e. extensions of the language]." (McGee [12], p. 58)

McGee sheds further light on the notion of open-endedness by linking what extensions of the language qualify as being legitimate to what individuals and classes of individual are nameable.[10] What he says can be summarized in the following rule:

McGee's Rule:

Consider a theory T formulated in a language L with at least one open-ended schema. Then:

(1) Any individual is nameable. If, for a given individual, L does not already contain a name for it, such a name can be added to L.

(2) Any collection of individuals C is nameable, in the sense that, if L does not already contain an open sentence Φ which holds exactly of the members of C, predicates (or other expressions) can be added to L that allow formulating a sentence that holds exactly of the members of C.

[9] Shaughan Lavine is another friend of open-ended schemas (see [8], [9], [10], [11]). The overall picture is the same, but the details of Lavine's and McGee's accounts of open-ended schemas differ.

[10] McGee [12], p. 59.

This is a meta-theoretic, rather than object-theoretic, rule. It accomplishes two things. First, it tells us how the object-language can be extended; and second, it tells us how new items of the language relate to the domain of the relevant theory. For example, if we are dealing with arithmetic, the rule informs us that any number is nameable, and that, likewise, so is any collection of numbers in that an open sentence Φ which holds exactly of the numbers in that collection can be added to the language. Furthermore, we are told that, if the current language lacks the resources to facilitate some naming, the requisite items can be added to the language.

McGee's Rule is crucial to open-ended schemas in at least two fundamental respects. First, it is needed to make sense of the idea of an open-ended schema.[11] Second, the meta-theoretic rule that determines what qualifies as legitimate extensions of the language must be powerful enough to deliver categoricity. Otherwise it would not be of any help to McGee and others who seek categoricity to support determinacy of truth-value. As we shall see in next section, McGee's Rule is powerful enough to deliver the goods.

Compared to first-order arithmetic, open-ended schema arithmetic does better because first-order arithmetic fails to fully capture at least one crucial aspect of our mastery of arithmetic, namely that we understand that the induction schema remains in force when we extend the language by adding new names and new predicate letters. Open-ended schema arithmetic also does better than second-order arithmetic, but for a different reason: open-ended schemas walk free of the charges about ontological commitment raised against second-order quantification in the last section. In particular, with respect to arithmetic, while endorsing the induction axiom of second-order arithmetic commits one to the existence of *sets* (or collections) of natural numbers, no such commitment is incurred by an adoption of the open-ended schema of induction. Individual induction axioms obtained by substituting an open sentence into the open-ended schema "are only committed to numbers", and so, the open-ended schema is "metaphysically benign".[12] Yet, it delivers exactly what the truth-value realist is after, i.e. categoricity and thus determinacy of truth-value.[13]

4 The unsavory ontological commitments of open-ended schemas

If open-ended schemas delivered the benefits of second-order axiomatizations without incurring their philosophical problems, they would really be quite something. However, as we will now argue, the supposed advantages of open-ended schemas over second-order logic are illusory. As far as ontological commitment goes, the presence of McGee's Rule puts open-ended schemas on a par with second-order quantification.

To support our contention we will argue that there is a sense in which open-ended schemas are as potent as second-order quantifiers. The range of the second-order quantifiers is the full power set of the first-order domain. The second-order variables take their semantic value among the arbitrary subsets of the first-order domain. Similarly, open sentences that can be substituted into open-ended schemas take their semantic value among, or are true of, the members of arbitrary subsets of the first-order domain. Hence, given McGee's Rule, for every instance of a second-order axiom, the open-ended schema delivers a corresponding individual axiom. To

[11] At least, McGee would insist that his is the best or most promising way to support the idea of a *genuinely* open-ended schema. We can see no other proposal that does the trick and does not somehow involve second-order quantifiers.

[12] McGee [12], p. 60.

[13] McGee [12], p. 62. We refer the reader to Lavine [11] and McGee [12] for the technical details.

see this, consider some instance of the relevant (second-order) axiom and suppose that C – a collection of members of the first-order domain – is the semantic value of the second-order variable for this instance. Then McGee's Rule tells us that C is nameable in that an open sentence Φ which is true exactly of the members of C can be added to the language. This open sentence can be substituted into the open-ended schema. For, after all, the whole point of taking schemas to be open-ended is to let them remain in force when the language is expanded.

Therefore, given McGee's Rule, whatever second-order axioms deliver, open-ended schemas will match. If it was ever mysterious how open-ended schemas get their potency to deliver categoricity results, this should be somewhat clearer now. What is also the case, however, is that open-ended schemas are as philosophically suspicious as second-order quantification is – at least with respect to ontological commitment.[14]

As we have seen, McGee maintains that an endorsement of open-ended schemas does not commit one to the existence of sets, or classes, of objects in the first-order domain – numbers in the case of arithmetic. There is a sense in which he is right, but, importantly, there is a more interesting sense in which he is wrong.

When we consider open-ended schema arithmetic, it should be granted that there is no commitment to sets of natural numbers at the level of the object-theory. The reason is this: open-ended schema arithmetic only has first-order quantifiers, and first-order variables take numbers as their semantic values. So, no set of natural numbers is the semantic value of any of the variables of the theory. This is the sense in which McGee is right about open-ended schemas not carrying any commitment to sets.

However, there is another – and more interesting – sense in which McGee is wrong. To see this we need to switch from focusing narrowly on the object-theory to focusing more broadly on the *whole* story: the object-theory and the meta-theory, and perhaps the meta-meta-theory, and so on. To do this we need to amend the Quinean criterion of ontological commitment in a straightforward way. According to the Quinean criterion, ontological commitment is theory-specific and determined intra-theoretically: the ontological commitments of a theory are given by the values of the theory's own bound variables. What we would like to suggest is that, for some purposes, this criterion is ill-equipped to capture what the relevant ontological commitments are. In some contexts, the relevant ontological commitments go "further up" – i.e. they go beyond the values of the bound variables of the object-theory.

It is a delicate and very difficult issue to say exactly in what contexts the modified criterion applies. We cannot address the issue fully here, but will indicate why the criterion should apply when we are dealing with theories with open-ended schemas in the context of the present discussion. The guiding thought is, roughly, that any principle which is indispensable to construing a theory in a certain way is ontologically committing when the theory is construed in that way. It should be noted that this is so whether the principle is part of the theory itself or appears

[14] It might be suggested that plural quantifiers can be invoked to understand open-ended schemas without incurring the ontological commitments of second-order quantification. Boolos has shown that monadic second-order quantification can be interpreted using plural constructions ([2] and [3]). While second-order quantifiers bring on a commitment to sets of whatever entities are in the first-order domain, friends of plural quantification maintain that plural quantifiers commit one only to the first-order entities themselves. Even if it is supposed that they are right, we maintain that there would be something slightly odd about appealing to plural quantification in order to make sense of the notion of open-endedness. For, if the attractiveness of plural quantifiers is rooted in their "ontological innocence", why not apply the plural approach in the object-theory? That is, one could adopt the second-order formulation of induction and interpret the second-order quantifier plurally, or simply use a plural quantifier directly to state the axiom of induction.

"higher up", at some meta-theoretical level.[15]

Note that what is being proposed is *not* that the ordinary Quinean criterion should be replaced as a criterion for same-level ontological commitment. The same-level ontological commitments of the object-theory as well as those of the meta-theoretic principles are determined by the Quinean criterion. Rather, what is being proposed is that the same-level ontological commitments of the object-theory and its full ontological commitments are not always the same. In certain cases, the full commitments of the object-theory are its same-level commitments *combined* with the commitments of one or more meta-theoretic principles. Which cases? The cases in which the meta-theoretic principle, or principles, are indispensable, in the sense that the object-theory would fail to achieve a set goal in their absence.

Let us apply this thought to the discussion at hand. Recall that friends of open-ended arithmetic highlight the categoricity of the theory and take it to support the theses that the subject-matter of the theory is a class of isomorphic structures – the standard models of arithmetic, in effect – and that every arithmetical statement has a determinate truth value. McGee's Rule is not part of open-ended arithmetic, but should, nonetheless, be taken to be ontologically committing when looking at open-ended arithmetic in the way just indicated. For the rule is indispensable when it comes to showing open-ended arithmetic to be categorical. Open-ended arithmetic cannot be shown to be categorical unless every class of individuals is nameable (in the sense defined in clause (2) of McGee's Rule above). McGee's Rule is the principle that ensures that this can be done. As such, the principle is indispensable to establishing categoricity and supporting determinacy of truth-value, its supposed philosophical corollary.

Applying the modified criterion of ontological commitment, McGee's Rule is thus ontologically committing when open-ended arithmetic is thought of as a categorical theory with certain philosophical ramifications – which is exactly the way it is thought of when compared to second-order arithmetic. Open-ended arithmetic – regarded in the manner indicated – is therefore not just committed to the numbers that serve as the values of the bound variables of the theory itself, but likewise to classes of these – indeed, to a class for any combination of numbers. Why? Because McGee's Rule involves a quantifier that ranges over arbitrary collections of the first-order domain: any collection of members of the first-order domain can be named.

At the meta-theoretic level, then, there is thus a quantifier whose bound variables take as their value entities which, when considered as values of object-theory variables of the second-order theory, were deemed an "unsavory ontological commitment". Pushing them one level up does not make them disappear. They might be at one remove from where they "used to be" (in the second-order theory), but they are still there. Indeed, as we have just argued, the commitment to these entities is needed in order to think of open-ended arithmetic as a categorical theory in order to support determinacy of truth-value – which is exactly what the friend of open-ended arithmetic does when comparing it to second-order arithmetic. In light of these considerations, we conclude that the ontological commitments of McGee's Rule go on the tally when the ontological commitments of open-ended arithmetic are determined. While the criterion of ontological commitment relied on in the argument just given does not conform to the letter of the Quinean criterion, it is plausibly regarded as conforming to its spirit. For, as suggested above, the modified criterion can be considered an extension of the Quinean criterion in that the ontological commitments of some theory T – construed in a certain way – is determined

[15] An anonymous referee rightly brought attention to the importance of the issue at hand. We develop, discuss, and defend the modified criterion of ontological commitment touched upon here in greater detail in 'All the way up! Commitment and the hierarchy', a work in progress.

by taking the values of the bound variables of T together with the values of the bound variables of whatever principles need to be true in order to construe T in this way.

It should be noted that our objection has force whether the initial worry about ontological commitment that open-ended schemas were supposed to avoid is conceived as being a worry about ontological commitment to a given – and additional – type of entity or a worry about ontological commitment on a grand scale. Whichever way the objection against second-order quantification is construed, open-ended schemas are in trouble if second-order quantification is: we have argued that they are as ontologically committing as second-order quantifiers.

5 Conclusion

To prevent a certain kind of misunderstanding, let us finish by explicitly stating what we take the range of our argument to be. We have *not* defended second-order logic against the objection that it is ontologically troublesome because it carries with it significant set-theoretic commitments. For all we have said, it may well be: we here remain neutral on this issue. Our point is, however, that by adopting open-ended schemas one is as ontologically committed to sets of the entities in the first-order domain as one is by adopting standard second-order quantification. This suffices to undermine McGee's claim that open-ended schemas do not incur the "unsavory ontological commitments" that second-order quantifiers do. Consequently, although open-ended schemas are sufficiently potent to deliver categoricity, they do not offer any philosophical advantage over second-order quantification as far as ontological commitment goes.

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